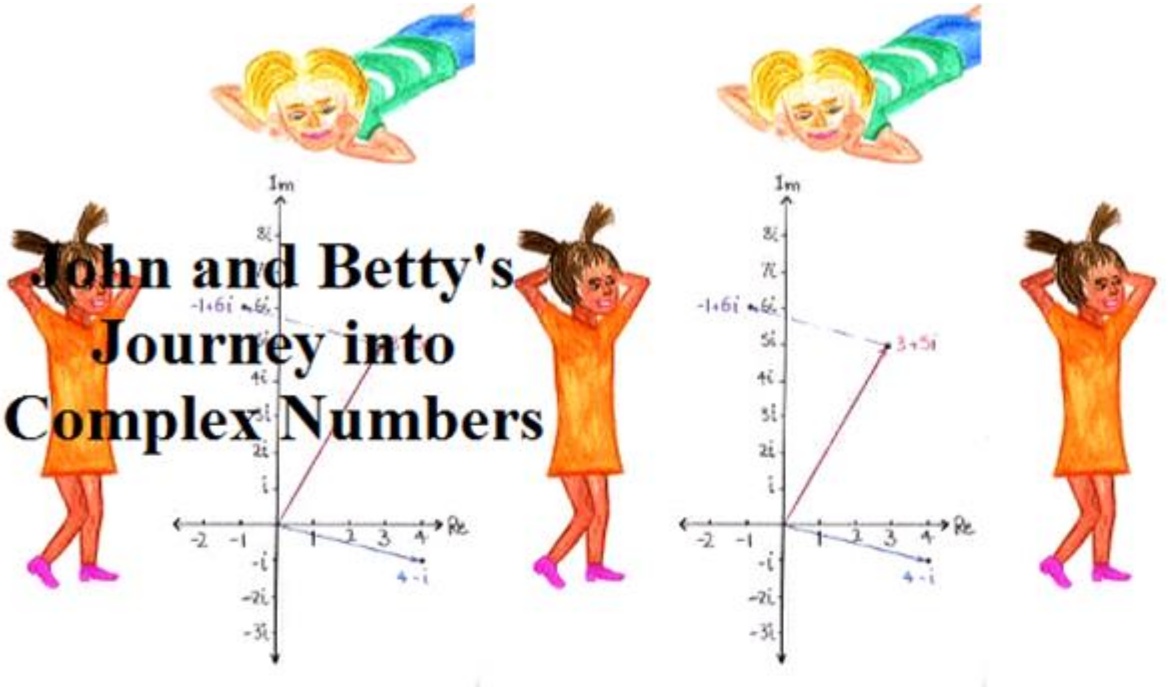
Further Maths Summer Work

- Topic: Complex numbers
- Read through these PowerPoint slides
- Answer the questions on the inserted slides
- Hand in first lesson back

By Matt Bower

John and Betty's Journey into Complex Numbers



John and Betty

Betty and John

One day John wanted to share 10 biscuits between Betty and himself.

"How many should we each get?" he asked Betty.

"Well, if we let x be the number of biscuits we each get then

$2x = 10$

$x = 10 \div 2$

$x = 5$

We should get 5 biscuits each," said Betty.

And she was right!

The next day John wanted to share his biscuits between Betty, himself and the dog, Trevor.

"How many should we each get this time?" he asked Betty.

Betty was puzzled for a while because no matter how they arranged the biscuits one of them ended up with an extra biscuit.

"We each get three, but there is always one biscuit left over."

"Wait!" said John. "What if we allow one of the biscuits to be broken up and each take one part out of three. So

$$3x = 10$$

$$x = 10 \div 3$$

$$x = 3\frac{1}{3}$$

We can have three and a third biscuits each!"

By fracturing the biscuits into pieces they had created a new sort of number called a "fraction". And it worked!

The next day Betty wanted to cut out a square piece of cardboard to make a sign.

"I want it to be exactly as big as three 1 metre square pieces." Betty exclaimed. "That's three square metres."

"How long should I make each side length?" Betty asked John.

John and Betty tried and tried to find the right number for the side length so the area of the square was three square metres but no whole numbers and no fractions worked .

They sat down with their heads in their hands. They were both totally discouraged.

"There must be a some length that can give us the right side length," huffed John. "But there aren't any whole numbers and there aren't any fractions. It's as though we'd have to make the number up if we wanted it to work."

Betty jumped up to her feet.

"That's it! Let's just make the number up!"

"Can we do that?" asked John.

"Why not?" said Betty. "Let's just make the side the right length so the square has area exactly three square metres, and call it the square root of three."

"I see," said John. "So if $\sqrt{3}$ means the square root of 3 then $\sqrt{3} \times \sqrt{3} = 3$".

"Exactly," replied Betty.

So they tried it.

And it worked!

Betty and John were both very happy. But Betty had one more question.

"John," asked Betty, "what if we wanted to find a number that when we times it by itself it equals -1?"

John thought this was a very interesting question. He thought about it for a long while.

He tried 1×1 but that equalled 1.

He tried -1×-1 but that also equalled 1.

He was stuck!

But then he thought about what Betty and he had done with the biscuits and done with the sign.

"I've got it!" he exclaimed. "Why don't we just make another number up?"

"What do you mean?" asked Betty.

"Well," said John slowly, "what if we make up a number, say 'i', so that

$$i \times i = -1."$$

"Can we do that?" queried Betty.

"Why not!" declared John.

"But there is no such number that has that size," said Betty.

"I know," replied John, "but the idea can exist in our imagination!" quipped John. "I think we should call it an *imaginary* number."

"Wow!" said Betty. "The concept is amazing. But what can we tell about this number 'i'? Is this the end of the journey, or is it the beginning?"

"I'm not sure," said John. "It depends on whether we have any questions."

"Well I have one," said Betty. "What would

$5i \times 3i$ be?"

"Why

$5 \times 3 \times i \times i = -15$,"

replied John.

And 5i + 3i would be 8i, chimed in Betty.

Now they were really starting to have some fun.

"Hey I wonder if we could mix up real numbers with imaginary numbers," asked John.

"What do you mean?" asked Betty.

"Well, we could say that 3 + 4i is a number."

"Yes," replied Betty, "but it has more parts to it than a normal number. What can we call it?"

"How about a complex number," giggled John.

And so they did.

"So what would $(3 - 5i) + (4 + 2i)$ equal?" asked Betty.

"Why, $7 - 3i$, of course," responded John. "But what about $(4-3i) \times (2+5i)$?"

"Well by expanding out the brackets,"

$$(4-3i) \times (2+5i) = 4 \times 2 + 4 \times 5i - 3i \times 2 - 3i \times 5i$$

$$= 8 + 20i - 6i - 15 \times -1$$

$$= 23 + 14i$$

said Betty, pretty excited about their new game.

Now try these

Simplify

- $(5 + 2i) + (8 +9i)$
- $(2 – i) – (-5 + 3i)$
- $3(8 – 4i)$
- $(2i)^3$
- Write in terms of *i*
- $\sqrt{(-169)}$
- $\sqrt{(-49)}$

Solve

- $x^2 + 2x + 5 = 0$
- $x^2 + 4x + 29 = 0$
- $x^2 - 6x + 18 = 0$

Simplify

- $(5 + i)(3 + 4i)$
- $(5 – 2i)(1 + 5i)$
- $(3 + 2i)(3-2i)$

"But what about (-4 + i)÷(3-2i) ?" quizzed Betty.

"Mmm. Maybe we could try

$$\frac{-4+i}{3-2i}\times\frac{3+2i}{3+2i} = \frac{-12+3i-8i+2\times i\times i}{9-6i+6i-2\times 2\times i\times i}$$

$$=\frac{-14-5i}{9+4}$$

$$= -\frac{14}{13}-\frac{5}{13}i$$

and look, it gives us a single complex number answer. Wow wee!" exclaimed John.

John and Betty were exhausted, so they rested for a while.

"One thing puzzles me Betty," mused John. "Where does 'i' go on the number line?"

A quizical look came over Betty's face. They tried putting it in all sorts of places on the number line but it just didn't fit. No matter where abouts they placed 'i' it was different to the number that was already there.

"It's as though our imaginary number 'i' isn't on the number line," said John. "But we must be able to put it somewhere. And what about 2i and 3i and -7i? We must be able to put them all somewhere."

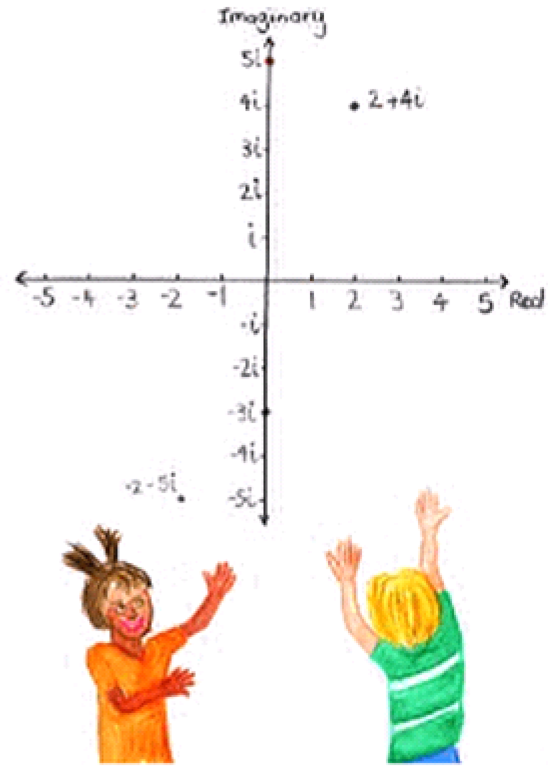
"What about giving imaginary numbers their own number line!" cried Betty.

"That's a fantastic idea!" cheered John with great gusto. "And I've got another great idea. Why don't we make our imaginary number line perpendicular to the real number line through the origin."

"What is the point of that?" asked Betty.

"Well," said John, "then not only will we have a place for imaginary numbers like 5i and -3i but also for complex numbers like 2+4i and -2 - 5i."

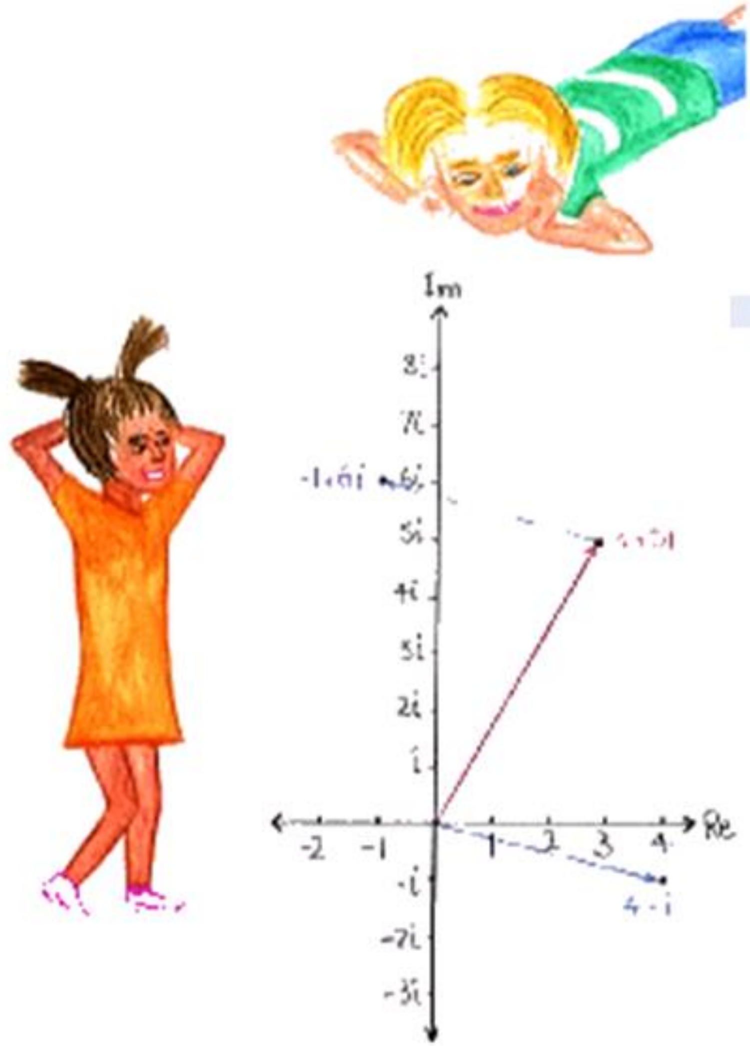
Betty was speechless. "That's brilliant!" she said. "Its like we now have a visual way for looking at complex numbers."



"What do you think our sums would look like on this complex number plane, Betty?"

"Well," said Betty, carefully considering the situation, "(3 + 5i) + (4 - i) would just be like placing the two lines formed by the numbers and the origin one after the other."

"Yes," said John. "And (3 + 5i) - (4 - i) would be like placing the line from 4 - i turned around after 3 + 5i."



Now try these:

- Simplify
- $(5 + 4i)/(2 - 3i)$
- $(25 - 10i)/(1 - 2i)$
- $(3 - 4i)^2/(1 + i)$
- Can you find an equation whose roots are $2 + 3i$ and $2 - 3i$?
- One root of a quadratic equation is $3-i$. What is the other root, and what was the equation?

Show these as vectors on an Argand diagram

- $7 + 2i$
- $5 - 4i$
- $3i + (6 - 4i)$
- $(2 + 7i) + (2 - 7i)$
- $(4 + 6i) - (3 +3i)$
- Roots of $x^2 -6x +10 = 0$
- Roots of $2x^2 +3x + 2 = 0$

Want to learn more?

- If so, read on at http://mathforum.org/mbower/johnandbetty/frame.htm
- But the remainder of the adventure is optional!
- **Please hand in the answers from the two slides on the first lesson back**

"But what about $(2+i)\times(3+2i)$? It seems a little bit trickier. It equals 4 + 7i. But can you notice any patterns John?"

John was very suspicious of the situation. "Look at the lengths of the three lines formed by our numbers from the origin, Betty.

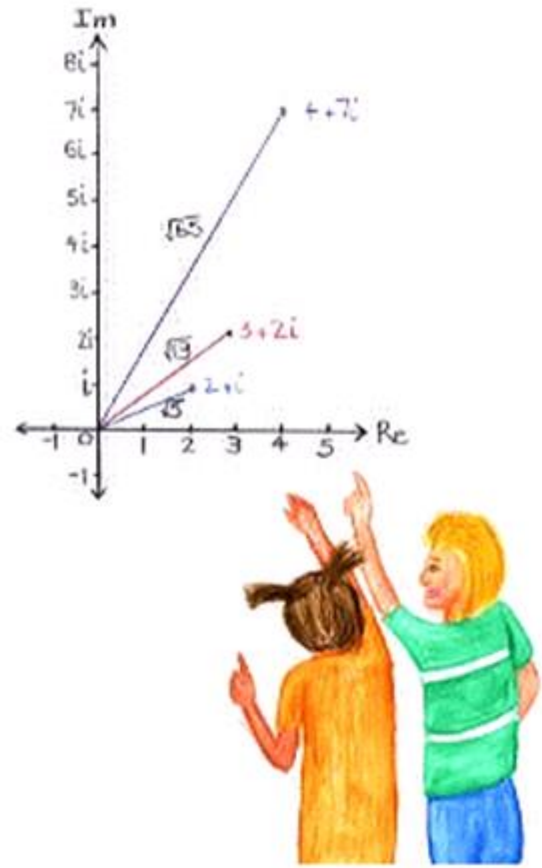
$2+i$ has length $\sqrt{5}$

$3 + 2i$ has length $\sqrt{13}$

and $4 + 7i$ has length $\sqrt{65}$.

"Oh, I see what you're saying John. The product of the length of the lines formed by the numbers we are multiplying equals the length of the result."

"Exactly Betty. But wait, there's more."



"Look at the angles formed by the lines. By using a protractor (or the tangent ratio)

2+i forms an angle of 0.464 radians (26° 34')

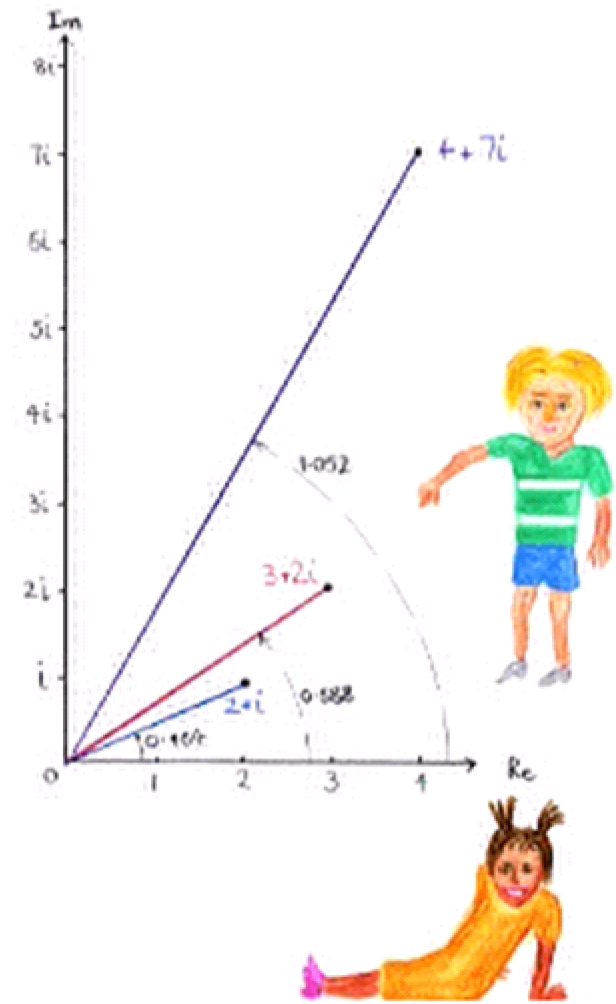
3+2i forms an angle of 0.588 radians (33° 41')

4 + 7i forms an angle of 1.052 radians (60°15')

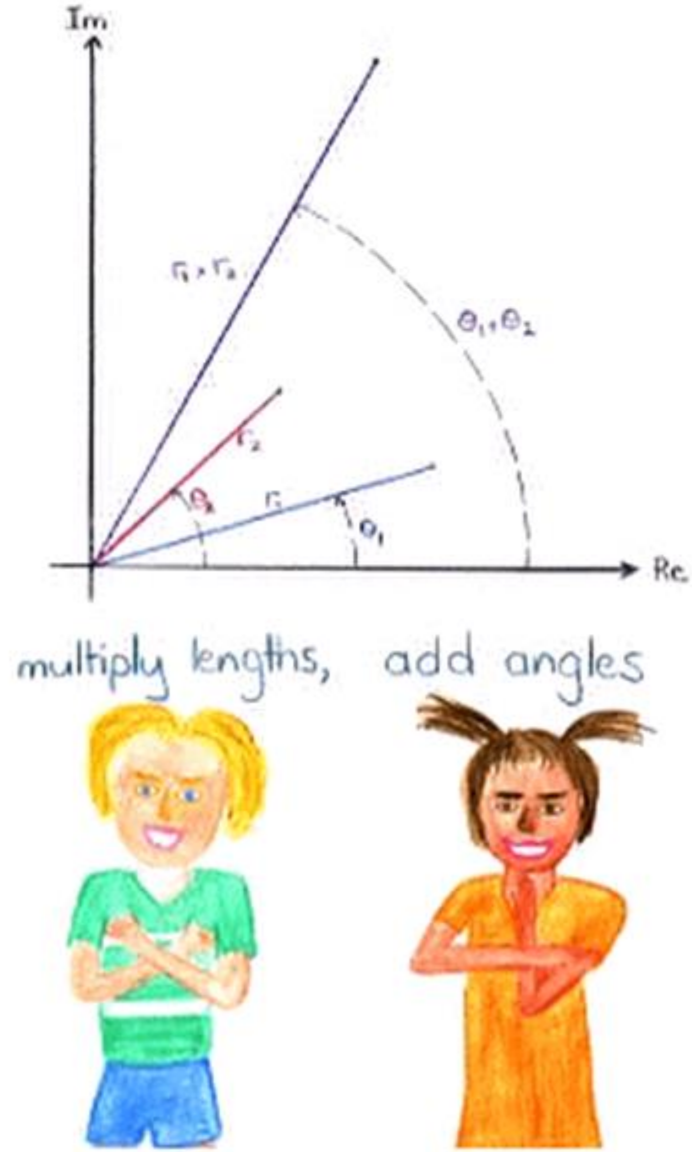
So the sum of the first two angles equals the angle of our result!"

"I wonder if it works for any other pairs of complex numbers, John."

They tried some more pairs of complex numbers. And do you know what - it worked for every single pair.



"So in a nutshell, John, if the product of 2 numbers is plotted on the complex number plane the result will always be at the place that is the sum of the angles (anticlockwise from the positive direction of the x-axis) and the product of the lengths."



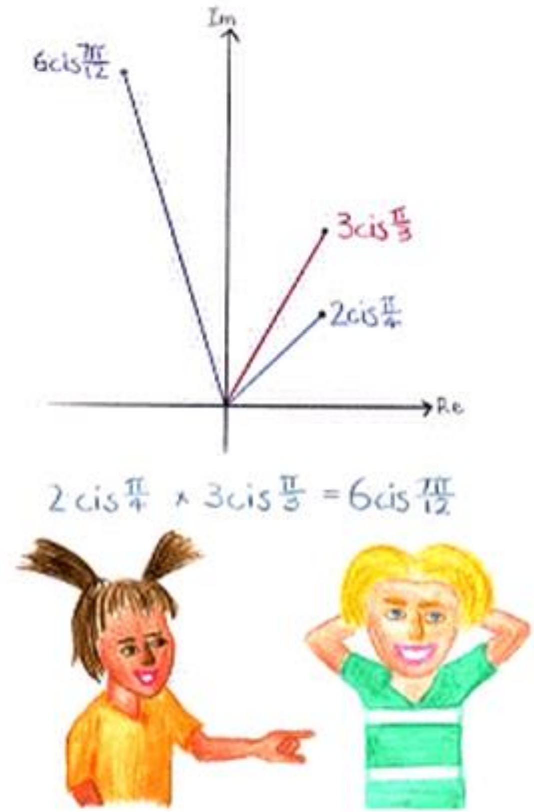
"That's it, Betty. So if that is the case Betty, why don't we make up a new way of identifying complex numbers. Why don't we call a complex number by the angle it forms (with the positive direction of the x-axis) and the length of the line drawn from the origin?"

"OK then John. Let's use radians because they are a more natural measure of angles. And let's use the notation "cis" so that $2\text{cis}\frac{\pi}{6}$ means the complex number on the plane that is 2 units from the origin at an angle of $\frac{\pi}{6}$."

"I see," said John, "so that

$$2\text{cis}\frac{\pi}{4} \times 3\text{cis}\frac{\pi}{3} = 6\text{cis}\frac{7\pi}{12} ."$$

"You've got it!", said Betty.



"There is only one thing that concerns me Betty. How can we relate the complex numbers using the 'cis' notation with the old way of naming complex numbers?"

"No problem," said Betty. "For a number like $2\text{cis}\frac{\pi}{6}$ we can see from the diagram that the real part is just $2\cos\frac{\pi}{6}$ and the imaginary part is $2\sin\frac{\pi}{6}$.

$$\text{So } 2\text{cis}\frac{\pi}{6} = 2\cos\frac{\pi}{6} + i\, 2\sin\frac{\pi}{6}$$

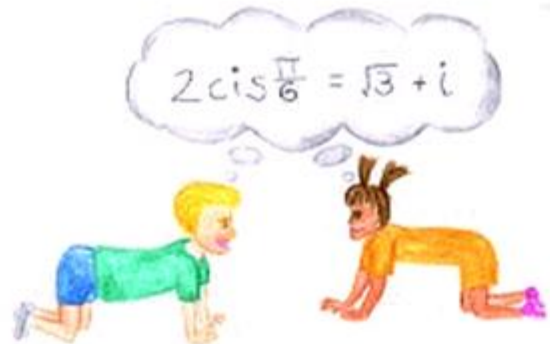
$$= \sqrt{3} + i$$

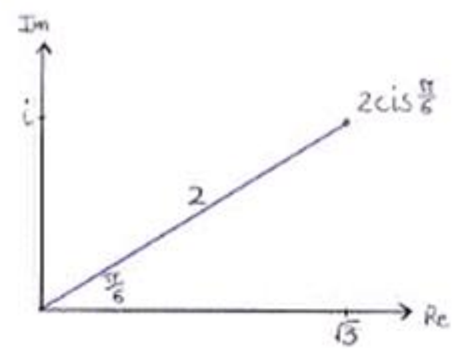
In fact we can say that in general

$$r\text{cis}\theta = r\cos\theta + i\, r\sin\theta$$

$$= r(\cos\theta + i\sin\theta)$$

"Yes!" exalted John.





"So what would it mean to cube a complex number?" asked Betty.

"That's easy", replied John. "For example

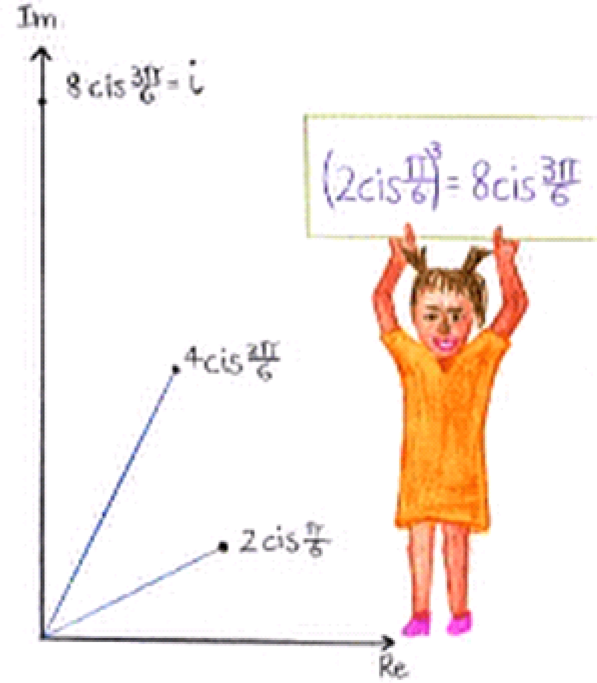
$$(2\text{cis}\tfrac{\pi}{6})^3 = 2\text{cis}\tfrac{\pi}{6} \times 2\text{cis}\tfrac{\pi}{6} \times 2\text{cis}\tfrac{\pi}{6}$$

$$= 4\text{cis}\tfrac{2\pi}{6} \times 2\text{cis}\tfrac{\pi}{6}$$

$$= 8\text{cis}\tfrac{3\pi}{6}$$

which actually equals 8i."

"Look at what happened," said Betty, all lit up from her discovery. "The length was cubed and the angle was multiplied by three. Do you think this suggests a pattern, John?"



"I can see it," chorused John. "It looks like

$$(r\,cis\,\theta)^n = r^n cis\, n\theta\ .\text{"}$$

And they tried it for every value of r, θ and n they could think of, and it worked every single time.

(This is called De Moivre's Theorem)

Both Betty and John were very tired, but they were also very pleased with what they had achieved. They had become so enthralled with the beauty of mathematics that they almost forgot about the world around them. And when they came out of their trance they looked at the world through different eyes.

Just then a beetle crawled onto Betty's shoe. "Isn't it beautiful!" they both exclaimed. The beetle was all of the colours of the rainbow.

"Look at how intricate and detailed the little legs are," said John. "I'd never looked at a beetle in so much detail before."

"Well," said Betty, "it just goes to show what can be discovered if you take the time to look."

The End